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THE COMPUTATION AND THE EXPANSION OF SOME TRIPLE
INTEGRALS ORIGINATING FROM THE THEORY OF COSMIC
RAYS

by

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R 261

The computation and the expansion of some triple integrals originating from the theory of cosmic rays.

Introduction.

In the theory of cosmic rays the following integrals occur

$$I(r, B) = r \int_{\Theta=0}^{\pi/2} (1-e^{-B\alpha})(1-e^{-B\beta})(1-e^{-B\gamma}) d\Theta \quad (0.0.1)$$

$$J^*(B) = B^{-5/2} \int_{r=0}^{\infty} I(r, B) dr \quad (0.0.2)$$

$$J(r) = \int_{B=0}^{\infty} B^{-5/2} I(r, B) dB \quad (0.0.3)$$

$$K = \int_{r=0}^{\infty} J(r) dr = \int_{B=0}^{\infty} J^*(B) dB \quad (0.0.4)$$

where α , β and γ are functions of r .

Two cases are considered here:

$$\text{first case: } \alpha = \frac{0.0245}{r^3} \quad (0.0.5)$$

$$\beta = \frac{0.0245}{r_1^3} \quad (0.0.6)$$

$$\gamma = \frac{0.0245}{r_2^3} \quad (0.0.7)$$

$$\text{second case: } \alpha = \frac{0.454}{r} (1+4r) e^{-4r^{2/3}} \quad (0.0.8)$$

$$\beta = \frac{0.454}{r_1} (1+4r_1) e^{-4r_1^{2/3}} \quad (0.0.9)$$

$$\gamma = \frac{0.454}{r_2} (1+4r_2) e^{-4r_2^{2/3}} \quad (0.0.10)$$

where $r_{1,2} = (a^2 + r^2 \pm 2ar \cos \Theta)^{1/2}$ and a denotes a positive constant.

In this report the behaviour of $J^*(B)$ for small positive and for large positive values of B is considered. Furthermore there are given numerical values of the functions $J(r)$, $J^*(B)$ and K for certain values of a in both cases.

CHAPTER I.

1st case:

1. Expansion of $J^*(B)$ for small positive values of B .

We split up the domain D of integration ($0 \leq r \leq \infty$, $0 \leq \theta \leq \frac{\pi}{2}$) into three domains:

$$D_1: 0 \leq \theta \leq \frac{\pi}{2} ; 0 \leq r \leq \frac{a}{2}$$

$$D_2: 0 \leq r_2 \leq B^{1/18}$$

and $D_3: D - D_1 - D_2$.

Let $j(b)$ be defined as the integral extended over the region D of the integrand $(1 - e^{-br_1^{-3}})(1 - e^{-br_2^{-3}})(1 - e^{-br_3^{-3}})$ with $r_{1,2} = (a^2 + r^2 \pm 2ar \cos \theta)^{\frac{1}{2}}$.

The integral over D_1 is called $h(b)$, that over D_2 is called $i(b)$ and finally that over D_3 is called $l(b)$, so that

$$j(b) = h(b) + i(b) + l(b).$$

2. Integration over the domain D_1 .

On D_1 we can expand the function $(1 - e^{-br_1^{-3}})(1 - e^{-br_2^{-3}})$ in a power-series of b , since on D_1 holds $r_1 \geq a$ and $r_2 \geq a/2$.

The function $h(b)$ appears to be equal to

$$b^2 \int_{r=0}^{\infty} r(1 - e^{-br^{-3}}) dr \int_{\theta=0}^{\frac{\pi}{2}} \left\{ (a^2 + r^2)^2 - 4a^2 r^2 \cos^2 \theta \right\}^{-\frac{3}{2}} d\theta + \dots \quad (1.2.1)$$

We know already:

$$\int_{\theta=0}^{\pi/2} \left\{ (a^2 + r^2)^2 - 4a^2 r^2 \cos^2 \theta \right\}^{-\frac{3}{2}} d\theta = (a^2 + r^2)^{-3} (1 - k^2)^{-1} E(k) \quad (1.2.2)$$

where $k = 2ar(a^2 + r^2)^{-1/2}$ and $E(k)$ denotes the elliptic function of the second kind;

if we substitute this result into (1.2.1) we find

$$h(b) = b^2 \int_{r=0}^{a/2} r(1 - e^{-br^{-3}}) (a^2 + r^2)^{-1} (a^2 + r^2)^{-2} E(k) dr + \dots \quad (1.2.3)$$

Substituting $u = b \cdot r^{-3}$, it follows that

$$h(b) = \frac{1}{3} b^{8/3} \int_{b(\frac{a}{2})^{-3}}^{\infty} \frac{u^{-5/3} (1-e^{-u}) \cdot E(k)}{(a^2+u^{-2/3}b^{2/3})(a^2-u^{-2/3}b^{2/3})^2} du + \dots \quad (1.2.4)$$

$$\text{with } k = \frac{2au^{-1/3} \cdot b^{1/3}}{a^2+u^{-2/3} \cdot b^{2/3}}.$$

This integral has to be expanded for small positive values of b . To this end we use the method described in Theorem 5 of MR 11 of the Computation Department of the Mathematical Centre. Then we find:

$$\begin{aligned} h(b) = & \frac{\pi}{6a^6} b^{8/3} \int_{u=0}^{\infty} u^{-5/3} (1-e^{-u}) du - \frac{\pi}{6a^6} b^{8/3} \int_{u=0}^{b(\frac{a}{2})^{-3}} u^{-5/3} (1-e^{-u}) du + \\ & + \frac{b^{8/3}}{3} \int_{u=b(\frac{a}{2})^{-3}}^{\infty} u^{-2/3} \left\{ \frac{E(k)}{(a^2-u^{-2/3}b^{2/3})^2(a^2+u^{-2/3}b^{2/3})} - \frac{\pi}{2a^6} \right\} du + \\ & + \frac{b^{8/3}}{3} \int_{u=b(\frac{a}{2})^{-3}}^{\infty} u^{-5/3} \{1-e^{-u}-u\} \left\{ \frac{E(k)}{(a^2-u^{-2/3}b^{2/3})^2(a^2+u^{-2/3}b^{2/3})} - \frac{\pi}{2a^6} \right\} du + \\ & \dots \quad (1.2.5) \end{aligned}$$

We will now compute the integrals occurring in (1.2.5).

The first and second term give the following results:

$$\frac{\pi}{6a^6} b^{8/3} \int_0^{\infty} u^{-5/3} (1-e^{-u}) du = - \frac{\pi}{6a^6} b^{8/3} \cdot \Gamma(-2/3) \quad (1.2.6)$$

and

$$- \frac{\pi}{6a^6} b^{8/3} \int_0^{b(\frac{a}{2})^{-3}} u^{-5/3} (1-e^{-u}) du = - \frac{\pi}{2a^6} b^3 \left(\frac{a}{2}\right)^{-1} + \dots \quad (1.2.7)$$

The third term can be computed by writing $r = (bu^{-1})^{1/3}$. This integral is then equal to:

$$b^3 \int_0^{a/2} r^{-2} \left\{ \frac{E(k)}{(a^2-r^2)^2(a^2+r^2)} - \frac{\pi}{2a^6} \right\} dr \quad (1.2.8)$$

The last integral does not depend on b , so the third term is of the order b^3 .

Using the expansions:

$$E(k) = \frac{\pi}{2} \left(1 - \frac{k^2}{2} + \dots\right)$$

and

$$(a^2 - r^2)^{-2} (a^2 + r^2) = a^{-6} \left(1 + \frac{r^2}{a^2} + \dots\right)$$

in the integrand of (1.2.8) one can derive that the third term is approximately equal to

$$0,0341\pi a^{-7} \cdot b^3 \quad (1.2.9)$$

From the relations (1.2.6), (1.2.7) and (1.2.9) we deduce:

$$h(b) = -\frac{\pi}{6a^6} \Gamma(-2/3) b^{8/3} + \dots \quad (1.2.10)$$

3. Integration over the domain D_2 .

In order to compute the contribution $i(b)$ we substitute $u = br_2^{-3}$ and find:

$$i(b) = \frac{2}{3} b^{8/3} \int_{u=b^{5/6}}^{\infty} Z(u) u^{-5/3} (1 - e^{-u}) du$$

with

$$Z(u) = \int_{r=R_1}^{r=R_2} \frac{r^{-2} dr}{\left\{2(a^2 + r^2) - u^{-2/3} b^{2/3}\right\}^{3/2} \left\{4a^2 r^2 - (a^2 + r^2 - u^{-2/3} b^{2/3})^2\right\}^{1/2}}$$

$$\text{with } R_1 = a - u^{-1/3} b^{1/3} \text{ and } R_2 = a + u^{-1/3} b^{1/3}. \quad (1.3.1.)$$

Using the substitutions $v = u^{-1/3} b^{1/3}$ and

$$r = a + pu^{-1/3} b^{1/3}$$

the equation (1.3.1) changes into the expression

$$i(b) = \frac{2}{3} b^{8/3} \int_{v=b^{5/6}}^{\infty} (1 - e^{-u}) u^{-5/3} f(v) \cdot dv \quad (1.3.2)$$

with

$$f(v) = v \int_{-1}^{+1} \frac{(a+pv)^{-2} dp}{\left\{2(2a^2 + 2apv + pv^2) - v^2\right\}^{3/2} \left\{4a^2(a+pv)^2 - [2a^2 + 2apv - (1-p^2)v^2]^2\right\}^{1/2}}$$

we have only to derive the first term of the power series of $f(v)$.

This term is found to be equal to

$$\begin{aligned}
f(v) &= 8^{-1} v \cdot a^{-5} \int_{-1}^{+1} \left\{ 4a^2(1-p^2)v^2 + 4p(1-p^2)av^3 - (1-p^2)^2 v^4 \right\}^{-\frac{1}{2}} dp + \dots = \\
&= 16^{-1} a^{-6} \int_{-1}^{+1} (1-p^2)^{-\frac{1}{2}} dp + \dots = \frac{\pi}{16} a^{-6} + \dots \quad (1.3.3)
\end{aligned}$$

By using the results of (1.2.6) and (1.3.3) we have

$$i(b) = - \frac{\pi}{24a^6} \Gamma(-2/3) b^{8/3} + \dots \quad (1.3.4)$$

Integration over D_3 :

In this domain we find

$$l(b) = b^3 \iint_{D_3} r_1^{-3} \cdot r_2^{-3} r^{-2} \cdot dr d\Theta + \dots \quad (1.4.1)$$

We have to remark that the remainder term is of the order b^3 , so that we need not to calculate terms of higher order occurring in the expansions for $h(b)$ and $i(b)$.

So the final result is:

$$j(b) = - \frac{5}{24} \frac{\pi}{a^6} \Gamma(-\frac{2}{3}) b^{8/3} + o(b^3) \quad (1.4.2)$$

6. Behaviour for large positive b :

The integration domain D is splitted up into D_4 ($0 \leq r \leq b^{1/3-\delta}$; $0 \leq \Theta \leq \frac{\pi}{2}$) and D_5 ($b^{1/3-\delta} \leq r \leq \infty$, $0 \leq \Theta \leq \frac{\pi}{2}$) with $\delta > 0$ but small with respect to $1/3$. Let I_4 be the integral extended over the region D_4 and I_5 be the integral extended over D_5 .

In the domain D_4 we have $1 - e^{-br^{-3}} \sim 1 + o(e^{-b^{3\delta}})$ and $1 - e^{-br_1^{-3}} \sim 1 + o(e^{-b^{3\delta}})$ for large positive values of b and consequently

$$I_4 \sim \frac{\pi}{4} b^{2/3-2\delta} \left\{ 1 + o(e^{-b^{3\delta}}) \right\} \quad (1.6.1)$$

In the domain D_5 we have $r_1 = r + o(b^{-1/3+\delta})$; $r_2 = r + o(b^{-1/3+\delta})$.

Hence

$$I_5 \sim \frac{\pi}{2} \int_{b^{1/3-\delta}}^{\infty} r(1 - e^{-br^{-3}})^3 dr + o(b^{-1+3\delta} e^{-b^{3\delta}}) \quad (1.6.2)$$

If we write now $u = br^{-3}$ we deduce from (1.6.2)

$$\begin{aligned} \frac{\pi}{2} \int_{r=b^{1/3-\delta}}^{\infty} r(1-e^{-br^{-3}})^3 dr &= -\frac{\pi}{6} b^{2/3} \Gamma(-2/3) \left[1 + 3^{1/3}(1-2^{2/3}) \right] 3^{2/3} - \\ &\quad - \frac{\pi}{6} b^{2/3} \int_{b^{3\delta}}^{\infty} u^{-5/3} (1-e^{-u})^3 du \end{aligned} \quad (1.6.3)$$

$$\begin{aligned} \text{Furthermore } \frac{\pi}{6} b^{2/3} \int_{b^{3\delta}}^{\infty} u^{-5/3} (1-e^{-u})^3 du &= - \\ &= -\frac{\pi}{4} b^{2/3-2\delta} \left\{ 1 + o(e^{-B^{3\delta}}) \right\}. \end{aligned}$$

By addition of I_4 and I_5 we find

$$j(b) \sim \frac{\pi}{6} 3^{2/3} \left\{ 1 + 3^{1/3} (1-2^{2/3}) \right\} \Gamma(-2/3) b^{2/3} + o(e^{-b^{3\delta}} b^{2/3-2\delta}), \quad (1.6.4)$$

CHAPTER II

2nd case:

Introduction:

In this Chapter we will give the order of the integral

$$R(B) = \int_{r=0}^{\infty} r dr \int_{\theta=0}^{\frac{\pi}{2}} (1-e^{-b\alpha})(1-e^{-b\beta})(1-e^{-b\gamma}) d\theta \quad (2.0.1)$$

for small and large positive values of b .

In (2.0.1) the functions α , β and γ are given by the relations:

$$\alpha = \frac{c}{r} (1 + 4r) e^{-4r^{2/3}} \quad (2.0.2)$$

$$\beta = \frac{c}{r_1} (1 + 4r_1) e^{-4r_1^{2/3}} \quad (2.0.3)$$

$$\gamma = \frac{c}{r_2} (1 + 4r_2) e^{-4r_2^{2/3}} \quad (2.0.4)$$

where c is a positive constant.

The integration domain D will again be splitted up into three domains:

$$D_1: 0 \leq r \leq a/2; 0 \leq \theta \leq \pi/2$$

$$D_2: 0 \leq r_2 \leq t \quad \text{with } t \text{ a positive constant}$$

$$D_3: D - D_1 - D_2 \quad .$$

Now the integral over D_1 is called $m(b)$, the integral over D_2 is called $n(b)$ and finally the integral over D_3 is called $p(b)$.

1. Estimation of $m(b)$ for small positive b .

In this domain we expand the function $(1-e^{-b\beta})(1-e^{-b\gamma})$ in a powerseries of b . Furthermore since

$$\left| \int_0^{\pi/2} \beta \cdot \gamma d\theta \right| \quad \text{is bounded} \quad \text{and} \quad |1-e^{-b\alpha}| < b\alpha$$

we find

$$|m(b)| < \frac{\pi}{2} c b^3 M \left| \int_0^{a/2} (1+4r) e^{-4r^{2/3}} dr \right| < b^3 M_1 \quad \text{with } M > 0$$

and $M_1 > 0$.

(2.1.1)

2. Treatment of the integral in the domain D_2 .

In D_2 we can estimate the integral by writing $(1-e^{-b\alpha})(1-e^{-b\beta}) = b^2\beta\alpha + \dots$, it follows then

$$n(b) = b^2 c^2 \iint_{D_2} (1+4r)(1+4r_1)r_1^{-1} e^{-4r_1^{2/3}} e^{-4r^{2/3}} (1-e^{-b\gamma}) dr d\theta + \dots \quad (2.2.1)$$

The function $(1+4r)(1+4r_1)r_1^{-1} e^{-4r_1^{2/3}} e^{-4r^{2/3}}$ is positive in D_2 and is furthermore bounded in D_2 .

Therefore we have

$$|n(b)| < M_2 b^2 c^2 \left| \iint_{D_2} (1-e^{-b\gamma}) d\theta dr \right| \quad \text{with } M_2 > 0. \quad (2.2.2)$$

If we now substitute $r = a \cos \theta + (a^2 \sin^2 \theta + r_2^2)^{\frac{1}{2}}$ we get finally

$$|n(b)| < M_2 b^2 c^2 \int_0^t r_2 (1-e^{-b\gamma}) \int_0^{bg \sin \frac{r_2}{a}} (a^2 \sin^2 \theta + r_2^2)^{-\frac{1}{2}} d\theta < M_4 b^3 c^3 \quad (2.2.3)$$

where M_4 is a positive constant.

3. The domain D_3 .

Now we can expand the integrand and we find easily

$$p(b) = b^3 c^3 \iint_{D_3} r\alpha \cdot \beta\gamma dr d\theta + \dots \quad (2.3.1)$$

According to (2.1.1), (2.2.3) and (2.3.1) it follows therefore:

$$b^{-5/2} R(b) = b^{\frac{1}{2}} O(1) + \dots \quad (2.3.2)$$

4. Behaviour for large B:

We divide the integration domain D into two parts. In D_1 and D_2 the variables r and θ satisfy the following inequalities:

$$D_1: 0 \leq r \leq b^{\varepsilon/2}; \quad 0 \leq \theta \leq \frac{\pi}{2} \quad \text{where } \varepsilon \text{ is a small positive number.}$$

$$D_2: D - D_1.$$

After writing $1-e^{-b\alpha} < 1$, it follows that the integral over D_1 is smaller than $\frac{\pi}{4} b^{\varepsilon}$. (2.4.1)

In the domain D_2 holds $r \sim r_1 \sim r_2$ for large positive b . Furthermore α , β and γ are exponentially small. So we can expand the exponential powers and we find the integral over D_2 to be equal to:

$$32 b^3 c^3 \pi \int_{b^{\varepsilon/2}}^{\infty} r e^{-12r^{2/3}} dr = \frac{2}{3} b^3 c^3 \pi (72b^{\frac{2\varepsilon}{3}} + 12 b^{\frac{\varepsilon}{3}+1}) e^{-12b^{\varepsilon/3}} \quad (2.4.2)$$

Combining (2.4.1) and (2.4.2) we find:

$$b^{-5/2} R(b) \sim b^{-5/2} O(1) . \quad (2.4.3)$$

5. Summary.

Combining the results of Chapter I and II we get in the first case after having substituted $b^* = 0,0245 b$ into 1.42 and 1.64

$$b^{-5/2} J^*(b) = 0,1332 10^{-3} a^{-6} b^{1/6} + O(b^{\frac{1}{2}}) . \quad (2.5.1)$$

for small positive b .

and

$$b^{-5/2} J^*(b) \sim 0,0564 b^{-11/6} + O(b^{-11/6-2\delta} e^{-b^{3\delta}}) . \quad (2.5.2)$$

for large positive b and with δ a small positive number.

For the second case we have according to (2.3.2) and (2.4.3)

$$b^{-5/2} R(b) = b^{\frac{1}{2}} O(1) \quad \text{for small positive } b. \quad (2.5.3)$$

and

$$b^{-5/2} R(b) = b^{-5/2} O(1) \quad \text{for large positive } b. \quad (2.5.4).$$

CHAPTER III

1. Method of computation.

In order to compute $J(r)$ and K according to (0.0.3) and (0.0.4) we change the triple integral into a double integral.

We define the function:

$$F(\alpha, \beta, \gamma) = \int_0^{\infty} B^{-5/2} (1-e^{-B\alpha})(1-e^{-B\beta})(1-e^{-B\gamma}) dB \quad (3.1.0)$$

Now we determine the derivatives with respect to α , β and γ . It follows

$$\frac{\partial F}{\partial \alpha} = \int_0^{\infty} B^{-3/2} e^{-B\alpha} (1-e^{-B\beta})(1-e^{-B\gamma}) dB \quad (3.1.1)$$

$$\frac{\partial^2 F}{\partial \alpha \partial \beta} = \int_0^{\infty} B^{-1/2} e^{-B\alpha} e^{-B\beta} (1-e^{-B\gamma}) dB \quad (3.1.2)$$

and

$$\frac{\partial^3 F}{\partial \alpha \partial \beta \partial \gamma} = \int_0^{\infty} B^{+1/2} e^{-B(\alpha+\beta+\gamma)} dB = \frac{1}{2} \sqrt{\pi} (\alpha+\beta+\gamma)^{-3/2} \quad (3.1.3)$$

$$\text{Furthermore holds } F(0, \beta, \gamma) = 0 \quad (3.1.4)$$

$$\frac{\partial F}{\partial \alpha} (\alpha, 0, \gamma) = 0 \quad (3.1.5)$$

$$\frac{\partial^2 F}{\partial \alpha \partial \beta} (\alpha, \beta, 0) = 0. \quad (3.1.6)$$

If we integrate the equation (3.1.3) three times and use the conditions (3.1.4), (3.1.5) and (3.1.6) we get finally

$$F(\alpha, \beta, \gamma) = \frac{4}{3} \sqrt{\pi} \left[-(\alpha+\beta+\gamma)^{3/2} + (\alpha+\beta)^{3/2} + (\alpha+\gamma)^{3/2} + (\beta+\gamma)^{3/2} - \alpha^{3/2} - \beta^{3/2} - \gamma^{3/2} \right] \quad (3.1.7)$$

According to (3.1.0) we can write $J(r)$ and K as a function of

$$J(r) = \int_0^{\pi/2} r F(\alpha, \beta, \gamma) d\theta \quad (3.1.8)$$

$$K = \int_0^{\infty} J(r) dr \quad (3.1.9)$$

The function $rF(\alpha, \beta, \gamma)$ was tabulated as a function of r and θ . The function $J(r)$ was obtained by numerical integration. In the same way the function $J^*(B)$ and the number K was defined for several a 's.

2. Numerical values.

The functions $J(r)$ and K were computed for the following values of a :

0.005; 0.01; 0.05; 0.1; 0.5; 1.

The function $J(B)$ was computed for $a = 0.1$ and $a = 1$.

<u>1st case</u>				<u>1st case</u>			
$a=0.005$				$a=0.01$			
r	$10^{-3}J(r)$	r	$10^{-3}J(r)$	r	$10^{-1}J(r)$	r	$10^{-1}J(r)$
0.0000	0	0.0110	24	0	0	0.0140	760
0.0005	129	0.0120	18	0.0005	585	0.0145	695
0.0010	238	0.0130	14	0.0010	1137	0.0150	632
0.0015	319	0.0140	11	0.0015	1648	0.0155	578
0.0020	365	0.0150	9	0.0020	2107	0.0160	529
0.0025	374	0.0160	7	0.0025	2497	0.0165	488
0.0030	354	0.0170	6	0.0030	2820	0.0170	447
0.0035	314	0.0180	5	0.0035	3062	0.0175	412
0.0040	269	0.0190	4	0.0040	3222	0.0180	379
0.0045	225	0.0200	4	0.0045	3303	0.0190	321
0.0050	186	0.0210	3	0.0050	3303	0.0200	278
0.0055	153	0.0220	3	0.0055	3247	0.0210	240
0.0060	126	0.0230	2	0.0060	3127	0.0220	209
0.0065	104	0.0240	2	0.0065	2971	0.0230	182
0.0070	86	0.0250	2	0.0070	2779	0.0240	159
0.0075	72	0.0260	1	0.0075	2582	0.0250	140
0.0080	60	0.0270	1	0.0080	2381	0.0300	77
0.0085	51	0.0280	1	0.0085	2178	0.0350	48
0.0090	43	0.0290	1	0.0090	1993	0.0400	30
0.0095	37	0.0300	0	0.0095	1816	0.0450	21
0.0100	31			0.0100	1650	0.0500	15
				0.0105	1491	0.0600	10
				0.0110	1354	0.0700	7
				0.0115	1226	0.0800	5
				0.0120	1115	0.0900	4
				0.0125	1006	0.1000	2
				0.0130	916	0.1100	1
				0.0135	834	0.1200	0

$K = 1895$

$K = 338$

1st case
a=0.05

r	J(r)	r	J(r)
0	.0	0.15	2.8
0.01	75.3	0.20	1.1
0.02	115.3	0.25	0.5
0.03	111.8	0.30	0.3
0.04	85.1	0.35	0.1
0.05	59.2	0.40	0.1
0.06	39.8	0.45	0.1
0.07	27.2	0.50	0
0.08	19.0		
0.09	13.6		
0.10	9.9		

K = 6.04

1st case
a=0.1

r	$10^{-1}J(r)$	r	$10^{-1}J(r)$
0	0	0.175	0.130
0.010	0.365	0.200	0.088
0.020	0.668	0.225	0.061
0.030	0.894	0.250	0.044
0.040	1.019	0.275	0.033
0.050	1.046	0.300	0.025
0.060	0.988	0.400	0.009
0.070	0.879	0.500	0.004
0.080	0.753	0.600	0.002
0.090	0.630	0.700	0.001
0.100	0.521	0.800	0.001
0.125	0.319	0.900	0
0.150	0.200		

K = 1.06

1st case
a=0.5

r	$10^2J(r)$	r	$10^2J(r)$
0	0	1.00	0.315
0.05	1.287	1.10	0.285
0.10	2.383	1.20	0.180
0.15	3.189	1.30	0.139
0.20	3.645	1.40	0.110
0.25	3.744	1.50	0.088
0.30	3.537	1.60	0.071
0.35	3.146	1.70	0.059
0.40	2.693	1.80	0.048
0.45	2.254	1.90	0.040
0.50	1.864	2.00	0.034
0.60	1.258	2.50	0.015
0.70	0.859	3.00	0.004
0.80	0.600	3.50	0.002
0.90	0.430	4.00	0.001

K = 0.0190

1st case
a=1

r	$10^3J(r)$	r	$10^3J(r)$
0	0	1.4	0.759
0.1	1.139	1.5	0.631
0.2	2.119	1.6	0.530
0.3	2.820	1.7	0.455
0.4	3.225	1.8	0.381
0.5	3.307	1.9	0.328
0.6	3.126	2.0	0.282
0.7	2.780	2.5	0.142
0.8	2.376	3.0	0.078
0.9	1.996	3.5	0.045
1.0	1.648	4.0	0.032
1.1	1.346	5.0	0.019
1.2	1.096	6.0	0
1.3	0.915		

K = 0.00336

1st case
a = 0.1

B	J [*] (B)
0	0
0.00005	25.24
0.00010	26.47
0.00025	28.17
0.00050	28.98
0.00100	29.41
0.00500	24.15
0.01000	18.10
0.02000	11.42
0.03000	7.955
0.04000	5.906
0.05000	4.597
0.06000	3.708
0.07000	3.070
0.08000	2.580
0.09000	2.226
0.10000	1.907
0.15000	1.066
0.20000	0.688
0.50000	0.157
1.00000	0.0480
5.00000	0.00278
10.00000	0.000797
20.00000	0.000227
50.00000	0.0000434

1st case
a = 1 .

B	J [*] (B)
0	0
0.025	0.678.10 ⁻⁴
0.050	0.748.10 ⁻⁴
0.075	0.785.10 ⁻⁴
0.100	0.811.10 ⁻⁴
0.200	0.886.10 ⁻⁴
0.300	0.928.10 ⁻⁴
0.400	0.936.10 ⁻⁴
0.500	0.950.10 ⁻⁴
1.000	0.973.10 ⁻⁴
3.000	0.879.10 ⁻⁴
5.000	0.770.10 ⁻⁴
10.000	0.576.10 ⁻⁴
20.000	0.3581.10 ⁻⁴
100.000	0.6052.10 ⁻⁵
1000.000	0.1516.10 ⁻⁶
10000.000	0.2524.10 ⁻⁸

2nd case
a=0.005

r	$10^{-1} \cdot J(r)$	r	$10^{-1} \cdot J(r)$
0.0000	0	0.0200	0.1653
0.0005	0.0846	0.0300	0.1253
0.0010	0.1460	0.0400	0.1013
0.0015	0.1953	0.0500	0.0852
0.0020	0.2340	0.0600	0.0737
0.0025	0.2664	0.0700	0.0648
0.0030	0.2905	0.0800	0.0579
0.0035	0.3103	0.0900	0.0520
0.0040	0.3267	0.1000	0.0471
0.0045	0.3381	0.2000	0.0226
0.0050	0.3561	0.3000	0.0134
0.0055	0.3346	0.4000	0.0084
0.0060	0.3264	0.5000	0.0055
0.0065	0.3125	0.6000	0.0037
0.0070	0.3008	0.7000	0.0026
0.0075	0.2934	0.8000	0.0018
0.0080	0.2835	1.0000	0.0009
0.0085	0.2756	1.2000	0.0005
0.0090	0.2678	1.4000	0.0003
0.0095	0.2601	1.6000	0.0001
0.0100	0.2529	1.8000	0.0000

K = 0.197

2nd case
a=0.01

r	J(r)	r	J(r)
0.000	0	0.0140	1.942
0.001	0.559	0.0150	1.879
0.002	0.967	0.0160	1.811
0.003	1.287	0.0170	1.759
0.004	1.545	0.0180	1.704
0.005	1.753	0.0190	1.654
0.006	1.915	0.0200	1.592
0.007	2.040	0.0250	1.393
0.008	2.132	0.0300	1.233
0.0082	2.148	0.0350	1.105
0.0084	2.162	0.0400	1.004
0.0086	2.176	0.0450	0.919
0.0088	2.191	0.0500	0.852
0.0090	2.203	0.0600	0.738
0.0092	2.215	0.0700	0.645
0.0094	2.224	0.0800	0.577
0.0096	2.231	0.0900	0.518
0.0098	2.241	0.1000	0.471
0.0100	2.259	0.2000	0.226
0.0102	2.230	0.3000	0.134
0.0104	2.216	0.4000	0.084
0.0106	2.201	0.5000	0.055
0.0108	2.188	0.6000	0.037
0.0110	2.170	0.7000	0.026
0.0112	2.154	0.8000	0.018
0.0114	2.137	1.0000	0.009
0.0116	2.122	1.2000	0.005
0.0118	2.105	1.4000	0.003
0.0120	2.090	1.6000	0.001
0.0130	2.017	1.8000	0.000

K = 0.183

2nd case
a=0.05

r	J(r)	r	J(r)
0	0	0.150	0.305
0.005	0.186	0.155	0.297
0.010	0.323	0.160	0.285
0.015	0.429	0.165	0.276
0.020	0.511	0.170	0.264
0.025	0.575	0.175	0.260
0.030	0.623	0.180	0.251
0.035	0.657	0.185	0.244
0.040	0.679	0.190	0.239
0.045	0.692	0.195	0.226
0.050	0.716	0.200	0.223
0.055	0.667	0.250	0.170
0.060	0.634	0.300	0.130
0.065	0.604	0.350	0.104
0.070	0.576	0.400	0.081
0.075	0.554	0.450	0.066
0.080	0.528	0.500	0.051
0.085	0.507	0.550	0.042
0.090	0.480	0.600	0.035
0.095	0.463	0.700	0.026
0.100	0.443	0.800	0.018
0.105	0.426	0.900	0.013
0.110	0.410	1.000	0.009
0.115	0.391	1.100	0.007
0.120	0.376	1.200	0.005
0.125	0.363	1.300	0.003
0.130	0.347	1.400	0.002
0.135	0.338	1.500	0.001
0.140	0.326	1.600	0.000
0.145	0.316		

K = 0.135

2nd case
a=0.1

r	J(r)
0	0
0.01	0.104
0.02	0.181
0.03	0.240
0.04	0.286
0.05	0.320
0.06	0.347
0.07	0.359
0.08	0.370
0.09	0.372
0.10	0.369
0.125	0.318
0.150	0.276
0.175	0.239
0.200	0.208
0.250	0.160
0.300	0.130
0.350	0.100
0.400	0.081
0.500	0.053
0.600	0.036
0.700	0.025
0.800	0.018
0.900	0.013
1.000	0.010
1.100	0.007
1.200	0.005
1.300	0.003
1.400	0.002
1.500	0.001
1.600	0.000

K = 0.103

2nd case
a=0.5

r	$10^{+1} \cdot J(r)$
0.00	0
0.05	0.1308
0.10	0.2305
0.15	0.3030
0.20	0.3498
0.25	0.3733
0.30	0.3773
0.35	0.3658
0.40	0.3439
0.45	0.3159
0.50	0.2876
0.60	0.2147
0.70	0.1596
0.80	0.1182
0.90	0.0874
1.00	0.0648
1.10	0.0481
1.20	0.0359
1.30	0.0267
1.40	0.0200
1.50	0.0151
1.60	0.0114
1.70	0.0086
1.80	0.0064
1.90	0.0050
2.00	0.0038
2.50	0.0010
3.00	0.0001
3.50	0.0000

K= 0.0245

2nd case
a=1

r	$10^{+2} \cdot J(r)$
0	0
0.1	0.228
0.2	0.406
0.3	0.528
0.4	0.588
0.5	0.595
0.6	0.562
0.7	0.496
0.8	0.423
0.9	0.349
1.0	0.285
1.1	0.222
1.2	0.174
1.3	0.136
1.4	0.106
1.5	0.082
1.6	0.063
1.7	0.050
1.8	0.039
1.9	0.031
2.0	0.024
2.5	0.007
3.0	0.002

K = 0.00549

2nd case
a = 0.1

B	J [*] (B)
0	0
0.005	0.0388
0.010	0.0493
0.030	0.0712
0.050	0.0791
0.100	0.0828
0.150	0.0809
0.200	0.0749
0.250	0.0691
0.350	0.0583
0.450	0.0496
0.650	0.0370
0.850	0.0287
1.050	0.0231
1.550	0.0147
2.050	0.0106
2.550	0.00800
3.550	0.00518
4.550	0.00369
5.550	0.00255
10.000	0.000996
20.000	0.000309
40.000	0.0000886

2nd case
a = 1

B	J [*] (B)
0	0
0.025	0.554.10 ⁻⁴
0.050	0.756.10 ⁻⁴
0.075	0.899.10 ⁻⁴
0.100	0.1013.10 ⁻³
0.200	0.1314.10 ⁻³
0.300	0.1505.10 ⁻³
0.400	0.1639.10 ⁻³
0.600	0.1815.10 ⁻³
0.800	0.1929.10 ⁻³
1.000	0.1998.10 ⁻³
1.600	0.2080.10 ⁻³
2.200	0.2076.10 ⁻³
2.800	0.2039.10 ⁻³
3.400	0.1973.10 ⁻³
4.000	0.1903.10 ⁻³
5.000	0.1778.10 ⁻³
6.000	0.1657.10 ⁻³
10.000	0.1256.10 ⁻³
20.000	0.0705.10 ⁻³